On the group of purely inseparable points of an abelian variety defined over a function field of positive characteristic

Damian RÖSSLER*

November 30, 2012

Abstract

Let K be the function field of a smooth and proper curve S over an algebraically closed field k of characteristic $p > 0$. Let A be an ordinary abelian variety over K. Suppose that the Néron model $\mathcal{A}$ of A over S has a closed fibre $\mathcal{A}_s$, which is an abelian variety of p-rank 0. We show that under these assumptions the group $A(K^{\rm perf})/{\rm Tr}_{K|k}(A)(k)$ is finitely generated. Here $K^{\rm perf} = K^{p^{-\infty}}$ is the maximal purely inseparable extension of K. This result implies that in some circumstances, the "full" Mordell-Lang conjecture, as well as a conjecture of Esnault and Langer, are verified.

1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$ and let S be a connected, smooth and proper curve over k. Let $K := \kappa(S)$ be its function field. If V/S is a locally free coherent sheaf on S, we denote by

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{{\rm hn}(V)} = V$$

*Institut de Mathématiques, Equipe Emile Picard, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse cedex 9, FRANCE, E-mail: rossler@math.univ-toulouse.fr

the Harder-Narasimhan filtration of V. We write as usual

$$\deg(*) := \deg(\mathrm{c}_1(*)),\ \mu(*) := \deg(*)/\mathrm{rk}(*)$$

and

$$\mu_{\min}(V) := \mu(V/V_{\mathrm{hn}(V)-1}),\ \mu_{\max}(V) := \mu(V_1).$$

See [2, chap. 5] (for instance) for the definition of the Harder-Narashimha filtration and for the notion of semistable sheaf, which underlies it.

A locally free sheaf V on S is said to be strongly semistable if $F_S^{r,*}(V)$ is semistable for all $r \in \mathbb{N}$. Here F_S is the absolute Frobenius endomorphism of S. A. Langer proved in [16, Th. 2.7, p. 259] that there is an $n_0 = n_0(V) \in \mathbb{N}$ such that the quotients of the Harder-Narasimhan filtration of $F_S^{n_0,*}(V)$ are all strongly semistable. This shows in particular that the following definitions :

$$\bar{\mu}_{\min}(V) := \lim_{l\to\infty} \mu_{\min}(F_S^{l*}(V))/p^l$$

and

$$\bar{\mu}_{\max}(V) := \lim_{l\to\infty} \mu_{\max}(F_S^{l*}(V))/p^l$$

make sense.

With these definitions in hand, we are now in a position to formulate the results that we are going to prove in the present text.

Let $\pi : \mathcal{A} \to S$ be a smooth commutative group scheme and let $A := \mathcal{A}_K$ be the generic fibre of $\mathcal{A}$. Let $\epsilon : S \to \mathcal{A}$ be the zero-section and let $\omega := \epsilon^*(\Omega^1_{\mathcal{A}/S})$ be the Hodge bundle of $\mathcal{A}$ over S.

Fix an algebraic closure $\bar{K}$ of K. For any $\ell \in \mathbb{N}$, let

$$K^{p^{-\ell}} := \{x \in \bar{K} | x^{p^l} \in K\},$$

which is a field. We may then define the field

$$K^{\mathrm{perf}} = K^{p^{-\infty}} = \cup_{\ell\in\mathbb{N}} K^{p^{-\ell}},$$

which is often called the *perfection* of K.

Theorem 1.1. *Suppose that $\mathcal{A}/S$ is semiabelian and that A is a principally polarized abelian variety. Suppose that the vector bundle ω is ample. Then there exists $\ell_0 \in \mathbb{N}$ such the natural injection $A(K^{p^{-\ell_0}}) \hookrightarrow A(K^{\mathrm{perf}})$ is surjective (and hence a bijection).*

For the notion of ampleness, see [12, par. 2]. A smooth commutative S-group scheme $\mathcal{A}$ as above is called semiabelian if each fibre of $\mathcal{A}$ is an extension of an abelian variety by a torus (see [6, I, def. 2.3] for more details).

We recall the following fact, which is proven in [1]: a vector bundle V on S is ample if and only if $\bar{\mu}_{\min}(V) > 0$.

Theorem 1.2. *Suppose that A is an ordinary abelian variety. Then*

(a) $\bar{\mu}_{\min}(\omega) \geqslant 0$*;*

(b) *if there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an abelian variety of p-rank 0, then $\bar{\mu}_{\min}(\omega) > 0$.*

Corollary 1.3. *Suppose that A is ordinary and that there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an abelian variety of p-rank 0. Then*

(a) *there exists $\ell_0 \in \mathbb{N}$ such the natural injection $A(K^{p^{-\ell_0}}) \hookrightarrow A(K^{\mathrm{perf}})$ is surjective;*

(b) *the group $A(K^{\mathrm{perf}})/\mathrm{Tr}_{K|k}(A)(k)$ is finitely generated.*

The notation $\mathrm{Tr}_{K|k}(A)$ refers to the $K|k$-trace of A over k. This is an abelian variety over k, which comes with a morphism $\mathrm{Tr}_{K|k}(A)_K \to A$. See [4] for the definition.

Here is an application of Corollary 1.3. Suppose until the end of the sentence that A is an elliptic curve over K and that $j(A) \not\in k$ (here $j(\cdot)$ is the modular j-invariant); then $\mathrm{Tr}_{K|k}(A) = 0$, A is ordinary and there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an elliptic curve of p-rank 0 (i.e. a supersingular elliptic curve); thus $A(K^{\mathrm{perf}})$ is finitely generated. This was also proven by D. Ghioca (see [7]) using a different method.

We list two further applications of Theorems 1.1 and 1.2.

Let Y be an integral closed subscheme of $B := A_{\bar{K}}$.

Let $C := \mathrm{Stab}(Y)^{\mathrm{red}}$, where $\mathrm{Stab}(Y) = \mathrm{Stab}_B(Y)$ is the translation stabilizer of Y. This is the closed subgroup scheme of B, which is characterized uniquely by the fact that for any scheme T and any morphism $b : T \to B$, translation by b on

the product $B \times_{\bar{K}} T$ maps the subscheme $Y \times T$ to itself if and only if b factors through $\mathrm{Stab}_B(Y)$. Its existence is proven in [10, exp. VIII, Ex. 6.5 (e)].

The following proposition is a special case of the (unproven) "full" Mordell-Lang conjecture, first formulated by Abramovich and Voloch. See [8] and [19, Conj. 4.2] for a formulation of the conjecture and further references.

Proposition 1.4. *Suppose that A is an ordinary abelian variety. Suppose that there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an abelian variety of p-rank 0. Suppose that $\mathrm{Tr}_{\bar{K}|k}(A) = 0$. If $Y \cap A(K^{\mathrm{perf}})$ is Zariski dense in Y then Y is the translate of an abelian subvariety of B by an point in $B(\bar{K})$.*

Proof (of Proposition 1.4). This is a direct consequence of Corollary 1.3 and of the Mordell-Lang conjecture over function fields of positive characteristic; see [13] for the latter. $\square$

Our second application is to a conjecture of a A. Langer and H. Esnault. See [5, Remark 6.3] for the latter. The following proposition is a special case of their conjecture.

Proposition 1.5. *Suppose that $k = \bar{\mathbb{F}}_p$. Suppose that A is an ordinary abelian variety and that there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an abelian variety of p-rank 0. Suppose that for all $\ell \geqslant 0$ we are given a point $P_\ell \in A^{(p^\ell)}(K)$ and suppose that for all $\ell \geqslant 1$, we have $\mathrm{Ver}^{(p^\ell)}_{A/K}(P_\ell) = P_{\ell-1}$. Then P_0 is a torsion point.*

Here $\mathrm{Ver}^{(p^\ell)}_{A/K} : A^{(p^\ell)} \to A^{(p^{\ell-1})}$ is the Verschiebung morphism. See [9, $\mathrm{VII_A}$, 4.3] for the definition.

Proof (of Proposition 1.5). By assumption the point P_0 is p^∞-divisible in $A(K^{\mathrm{perf}})$, because $[p]_{A/K} = \mathrm{Ver}_{A/K} \circ \mathrm{Frob}_{A/K}$ and $\mathrm{Ver}_{A/K}$ is étale, because A is ordinary. Here $\mathrm{Frob}_{A/K}$ is the relative Frobenius morphism and $[p]_{A/K}$ is the multiplication by p morphism on A. Thus the image of P_0 in $A(K^{\mathrm{perf}})/\mathrm{Tr}_{K|k}(A)(k)$ is a torsion point because the group $A(K^{\mathrm{perf}})/\mathrm{Tr}_{K|k}(A)(k)$ is finitely generated by Corollary 1.3. Hence P_0 is a torsion point because $\mathrm{Tr}_{K|k}(A)(k)$ consists of torsion points. $\square$

Acknowledgments. I would like to thank H. Esnault and A. Langer for many interesting conversations on the subject matter of this article.

2 Proof of 1.1, 1.2 & 1.3

2.1 Proof of Theorem 1.1

The idea behind the proof of Theorem 1.1 comes from an article of M. Kim (see [15]).

In this subsection, the assumptions of Theorem 1.1 hold. So we suppose that $\mathcal{A}/S$ is semiabelian, that A is a principally polarized abelian variety and that ω is ample.

If $Z \to W$ is a W-scheme and W is a scheme of characteristic p, then for any $n \geqslant 0$ we shall write $Z^{[n]} \to W$ for the W-scheme given by the composition of arrows

$$Z \to W \stackrel{F_W^n}{\to} W.$$

Now fix $n \geqslant 1$ and suppose that $A(K^{p^{-n}})\backslash A(K^{p^{-n+1}}) \neq \emptyset$.

Fix $P \in A^{(p^n)}(K)\backslash A^{(p^{n-1})}(K) = A(K^{p^{-n}})\backslash A(K^{p^{-n+1}})$. The point P corresponds to a commutative diagram of k-schemes

$$\begin{tikzcd} & A \arrow[d] \\ \mathrm{Spec}\ K^{[n]} \arrow[ur, "P"] \arrow[r, "F_K^n"'] & \mathrm{Spec}\ K \end{tikzcd}$$

such that the residue field extension $K|\kappa(P(\mathrm{Spec}\ K^{[n]}))$ is of degree 1 (in other words P is birational onto its image). In particular, the map of K-vector spaces $P^*\Omega^1_{A/k} \to \Omega^1_{K^{[n]}/k}$ arising from the diagram is non zero.

Now recall that there is a canonical exact sequence

$$0 \to \pi_K^*\Omega^1_{K/k} \to \Omega^1_{A/k} \to \Omega^1_{A/K} \to 0.$$

Furthermore the map $F_K^{n,*}\Omega^1_{K/k} \stackrel{F_K^{n,*}}{\to} \Omega^1_{K^{[n]}/k}$ vanishes. Also, we have a canonical identification $\Omega^1_{A/K} = \pi_K^*\omega_K$ (see [3, chap. 4., Prop. 2]). Thus the natural surjection $P^*\Omega^1_{A/k} \to \Omega^1_{K^{[n]}/k}$ gives rise to a non-zero map

$$\phi_n : F_K^{n,*}\omega_K \to \Omega^1_{K^{[n]}/k}.$$

The next lemma examines the poles of the morphism ϕ_n.

We let E be the closed subset, which is the union of the points $s \in S$, such that the fibre $\mathcal{A}_s$ is not complete.

Lemma 2.1. *The morphism ϕ_n extends to a morphism of vector bundles*

$$F_S^{n,*}\omega \to \Omega^1_{S^{[n]}/k}(E).$$

Proof (of 2.1). First notice that there is a natural identification $\Omega^1_{S^{[n]}/k}(\log E) = \Omega^1_{S^{[n]}/k}(E)$, because there is a sequence of coherent sheaves

$$0 \to \Omega_{S^{[n]}/k} \to \Omega^1_{S^{[n]}/k}(\log E) \to \mathcal{O}_E \to 0$$

where the morphism onto $\mathcal{O}_E$ is the residue morphism. Here the sheaf $\Omega^1_{S^{[n]}/k}(\log E)$ is the sheaf of differentials on $S^{[n]}\backslash E$ with logarithmic singularities along E. See [14, Intro.] for this result and more details on these notions.

Now notice that in our proof of Theorem 1.1, we may replace K by a finite extension field K' without restriction of generality. We may thus suppose that A is endowed with an m-level structure for some $m \geqslant 3$.

We now quote part of one the main results of the book [6]:

(1) there exists a regular moduli space $A_{g,m}$ for principally polarized abelian varieties over k endowed with an m-level structure;

(2) there exists an open immersion $A_{g,m} \hookrightarrow A^*_{g,m}$, such that the (reduced) complement $D := A^*_{g,m}\backslash A_{g,m}$ is a divisor with normal crossings and $A^*_{g,m}$ is regular and proper over k;

(3) the scheme $A^*_{g,m}$ carries a semiabelian scheme G extending the universal abelian scheme $f : Y \to A_{g,m}$;

(4) there exists a regular and proper $A^*_{g,m}$-scheme $\bar{f} : \bar{Y} \to A^*_{g,m}$, which extends Y and such that $F := \bar{Y}\backslash Y$ is a divisor with normal crossings (over k); furthermore

(5) on $\bar{Y}$ there is an exact sequence of locally free sheaves

$$0 \to \bar{f}^*\Omega^1_{A_{g,m}/k}(\log D) \to \Omega^1_{Y/k}(\log F) \to \Omega^1_{Y/A_{g,m}}(\log F/D) \to 0,$$

which extends the usual sequence of locally free sheaves

$$0 \to f^*\Omega^1_{A_{g,m}/k} \to \Omega^1_{Y/k} \to \Omega^1_{Y/A_{g,m}} \to 0$$

on $A_{g,m}$. Furthermore there is an isomorphism $\Omega^1_{\bar{Y}/A_{g,m}}(\log F/D) \simeq \bar{f}^*\omega_G$. Here $\omega_G := \mathrm{Lie}(G)^\vee$ is the tangent bundle (relative to $A^*_{g,m}$) of G restricted to $A^*_{g,m}$ via the unit section.

See [6, chap. VI, th. 1.1] for the proof.

The datum of A/K and its level structure induces a morphism $\phi : K \to A_{g,m}$, such that $\phi^*Y \simeq A$, where the isomorphism respects the level structures. Call $\lambda : A \to Y$ the corresponding morphism over k. Let $\bar{\phi} : S \to A^*_{g,m}$ be the morphism obtained from ϕ via the valuative criterion of properness. By the unicity of semiabelian models (see [6, chap. I, th. I.9]), we have a natural isomorphism $\bar{\phi}^*G \simeq \mathcal{A}$ and thus we have a set-theoretic equality $\bar{\phi}^{-1}(D) = E$ and an isomorphism $\bar{\phi}^*\omega_G = \omega$. Let also $\bar{P}$ be the morphism $S^{[n]} \to \bar{Y}$ obtained from $\lambda \circ P$ via the valuative criterion of properness. By construction we now get an arrow

$$\bar{P}^*\Omega^1_{\bar{Y}/k}(\log F) \to \Omega^1_{S^{[n]}/k}(\log E)$$

and since the induced arrow

$$\bar{P}^*\bar{f}^*\Omega^1_{A_{g,m}/k}(\log D) = F_S^{n,*} \circ \bar{\phi}^*(\Omega^1_{A_{g,m}/k}(\log D)) \to \Omega^1_{S^{[n]}/k}(\log E)$$

vanishes (because it vanishes generically), we get an arrow

$$\bar{P}^*\Omega^1_{\bar{Y}/A^*_{g,m}}(\log F/D) = F_S^{n,*} \circ \bar{\phi}^*\omega_G = F_S^{n,*}\omega \to \Omega^1_{S^{[n]}/k}(\log E) = \Omega^1_{S^{[n]}/k}(E),$$

which is what we sought. $\square$

To conclude the proof of Proposition 1.1, choose l_0 large enough so that

$$\mu_{\min}(F_S^{l,*}(\omega)) > \mu(\Omega^1_{S/k}(E))$$

for all $l > l_0$. Such an l_0 exists because $\bar{\mu}_{\min}(\omega) > 0$. Now notice that since k is a perfect field, we have $\Omega^1_{S/k}(E) \simeq \Omega^1_{S^{[n]}/k}(E)$. We see that we thus have

$$\mathrm{Hom}(F_S^{l,*}(\omega), \Omega^1_{S^{[n]}/k}(E)) = 0$$

for all $l > l_0$ and thus by Lemma 2.1 we must have $n < l_0 + 1$. Thus we have

$$A(K^{(p^{-l})}) = A(K^{(p^{-l+1})})$$

for all $l \geqslant l_0$.

Remark. The fact that $\mathrm{Hom}(F_S^{l,*}(\omega), \Omega^1_{S^{[n]}/k}(E)) \simeq \mathrm{Hom}(F_S^{l,*}(\omega), \Omega^1_{S/k}(E))$ vanishes for large l can also be proven without appealing to the Harder-Narasimhan filtration. Indeed the vector bundle ω is also cohomologically p-ample (see [17, Rem. 6), p. 91]) and thus there is an $l_0 \in \mathbb{N}$ such that for all $l > l_0$

$$\begin{aligned}
\mathrm{Hom}(F_S^{l,*}(\omega), \Omega^1_{S/k}(E)) &= H^0(S, F_S^{l,*}(\omega)^\vee \otimes \Omega^1_{S/k}(E)) \\
&\overset{\text{Serre duality}}{=} H^1(S, F_S^{l,*}(\omega) \otimes \Omega^1_{S/k}(E)^\vee \otimes \Omega^1_{S/k})^\vee \\
&= H^1(S, F_S^{l,*}(\omega) \otimes \mathcal{O}(-E))^\vee = 0.
\end{aligned}$$

2.2 Proof of Theorem 1.2

In this subsection, we suppose that the assumptions of Theorem 1.2 hold. So we suppose that A is an ordinary abelian variety.

Notice first that for any $n \geqslant 0$, the Hodge bundle of $\mathcal{A}^{(p^n)}$ is $F_S^{n,*}\omega$. Hence, in proving Proposition 1.2, we may assume without restriction of generality that ω has a strongly semistable Harder-Narasimhan filtration.

Let $V := \omega/\omega_{\mathrm{hn}(\omega)-1}$. Notice that for any $n \geqslant 0$, we have a (composition of) Verschiebung(s) map(s) $\omega \to F_S^{n,*}\omega$. Composing this with the natural quotient map, we get a map

$$\phi : \omega \overset{\mathrm{Ver}_{\mathcal{A}}^{(p^n),*}}{\to} F_S^{n,*}V \tag{1}$$

The map ϕ is generically surjective, because by the assumption of ordinariness the map $\omega \overset{\mathrm{Ver}^{(p^n),*}}{\to} F_S^{n,*}\omega$ is generically an isomorphism.

We now prove (a). The proof is by contradiction. Suppose that $\bar{\mu}_{\min}(\omega) := \mu(V) < 0$. This implies that when $n \to \infty$, we have $\mu(F_S^{n,*}V) \to -\infty$. Hence if n is sufficiently large, we have $\mathrm{Hom}(\omega, F_S^{n,*}V) = 0$, which contradicts the surjectivity of the map in (1).

We turn to the proof of (b). Again the proof is by contradiction. So suppose that $\bar{\mu}_{\min}(\omega) \leqslant 0$. By (a), we know that we then actually have $\bar{\mu}_{\min}(\omega) = 0 = \mu(V)$ and $V \neq 0$. If $\bar{\mu}_{\max}(\omega) > 0$ then the map $\omega_1 \to F_S^{n,*}V$ obtained by composing ϕ with the inclusion $\omega_1 \hookrightarrow \omega$ must vanish, because

$$\mu(\omega_1) > \mu(F_S^{n,*}V) = p^n \cdot \mu(V) = 0.$$

Hence we obtain a map $\omega/\omega_1 \to F_S^{n,*}V$. Repeating this reasoning for ω/ω_1 and applying induction we finally get a map

$$\lambda : V \to F_S^{n,*}V.$$

The map λ is generically surjective and thus globally injective, since its target and source are locally free sheaves of the same generic rank. Let T be the cokernel of λ (which is a torsion sheaf). We then have

$$\deg(V) + \deg(T) = 0 + \deg(T) = \deg(F_S^{n,*}V) = 0$$

and thus $T = 0$. This shows that λ is a (global) isomorphism. In particular, the map ϕ is surjective. Thus the map

$$\phi_s : \omega_s \overset{\mathrm{Ver}_{\mathcal{A}_s}^{(p^n),*}}{\to} F_s^{n,*}V_s$$

is surjective and thus non-vanishing. This contradicts the hypothesis on the p-rank at s.

2.3 Proof of Corollary 1.3

In this subsection, we suppose that the assumptions of Corollary 1.3 are satisfied. So we suppose that A is ordinary and that there is a closed point $s \in S$ such that $\mathcal{A}_s$ is an abelian variety of p-rank 0.

We first prove (a). First we may suppose without restriction of generality that A is principally polarized. This follows from the fact that the abelian variety $(A \times_K A^\vee)^4$ carries a principal polarization ("Zarhin's trick" - see [18, Rem. 16.12, p. 136]) and from the fact that the abelian variety $(A \times_K A^\vee)^4$ also satisfies the assumptions of Corollary 1.3. Furthermore, we may without restriction of generality replace S by a finite extension S'. Thus, by Grothendieck's semiabelian reduction theorem (see [11, IX]) we may assume that $\mathcal{A}$ is semiabelian. Statement (a) then follows from Theorems 1.1 and 1.2.

We now turn to statement (b). Let $\tau_{K|k} : \mathrm{Tr}_{K|k}(A)_K \to A$ be the $K|k$-trace morphism. Notice that for any $\ell \geqslant 0$, we have a natural identification of k-group schemes $\mathrm{Tr}_{K|k}(A)^{(p^\ell)} \simeq \mathrm{Tr}_{K|k}(A^{(p^\ell)})$, because the extension K/K^p is primary and k is perfect (see [4, Th. 6.4 (3)]). Thus, if $\ell_0 \in \mathbb{N}$ is the number appearing in (a),

we have identifications

$$\begin{aligned} A(K^{\mathrm{perf}})/\mathrm{Tr}_{K|k}(A)(k) &= A(K^{-\ell_0})/\mathrm{Tr}_{K|k}(A)(k) = A(K^{-\ell_0})/\mathrm{Tr}_{K|k}(A)(k^{-\ell_0}) \\ &= A^{(p^{\ell_0})}(K)/\mathrm{Tr}_{K|k}(A)^{(p^{\ell_0})}(k) = A^{(p^{\ell_0})}(K)/\mathrm{Tr}_{K|k}(A^{(p^{\ell_0})})(k) \end{aligned}$$

and the group appearing after the last equality is finitely generated by the Lang-Néron theorem.

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